

ℓ -DISTANCE-BALANCED GRAPHS

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Abstract

Let ℓ denote a positive integer. A connected graph Γ of diameter at least ℓ is said to be ℓ -distance-balanced whenever for any pair of vertices u, v of Γ such that $d(u, v) = \ell$, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u . In this paper we present some basic properties of ℓ -distance-balanced graphs and study in more detail ℓ -distance-balanced graphs of diameter at most 3. We also investigate the ℓ -distance-balanced property of some well known families of graphs such as the generalized Petersen graphs.

1 Introduction

Throughout this paper, all graphs are simple (without loops and multiple edges), undirected, finite and connected. Given a graph Γ let $V(\Gamma)$ and $E(\Gamma)$ denote its vertex set and edge set, respectively. For $u, v \in V(\Gamma)$ we denote the distance between u and v in Γ by $d_\Gamma(u, v)$ (or simply $d(u, v)$ if the graph Γ is clear from the context). Furthermore, for any nonnegative integer i and $u \in V(\Gamma)$ let $N_i(u) = \{v \in V(\Gamma) \mid d(u, v) = i\}$ (we abbreviate $N(u) = N_1(u)$). For $S \subseteq V(\Gamma)$ the subgraph of Γ induced by S is denoted by $\langle S \rangle$ (we abbreviate $\Gamma - S = \langle V(\Gamma) \setminus S \rangle$).

For any pair of vertices $u, v \in V(\Gamma)$ we let W_{uv} be the set of vertices of Γ that are closer to u than to v , that is

$$W_{uv} = \{w \in V(\Gamma) \mid d(u, w) < d(v, w)\}.$$

The pair u, v is said to be *balanced* if $|W_{uv}| = |W_{vu}|$ and is *non-balanced* otherwise. Let ℓ denote a positive integer. A connected graph Γ of diameter at least ℓ is said to be ℓ -distance-balanced whenever any pair of vertices $u, v \in V(\Gamma)$ at distance ℓ is balanced, that is, if for any $u, v \in V(\Gamma)$ such that $d(u, v) = \ell$ we have

$$|W_{uv}| = |W_{vu}|.$$

¹The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0285 and research projects N1-0032, N1-0038, J1-6720, J1-7051).

²The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0285 and research projects N1-0038, J1-6720, J1-7051).

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A connected graph Γ is said to be *highly distance-balanced* if it is ℓ -distance-balanced for every $1 \leq \ell \leq D$, where D is the diameter of Γ .

The ℓ -distance-balanced graphs are a natural generalization of the so-called *distance-balanced* graphs [10]. They were first defined by Boštjan Frelih in his PhD dissertation [6], where 2-distance-balanced graphs were studied in more detail. In particular, 2-distance-balanced graphs which are not 2-connected were characterized, and 2-distance-balanced graphs were studied with respect to various graph products. The ℓ -distance-balanced graphs were also the main topic of the paper [5]. However, some of the stated results do not hold while some are given without proof. We comment on two of these problems later (see Remarks 2.3 and 3.3).

On the other hand, distance-balanced graphs have been extensively studied, see [1, 7, 8, 9, 10, 11, 12, 13, 14]. We also point out that every distance-regular graph [3] is highly distance-balanced. The opposite is of course not true. For instance, the generalized Petersen graph $GP(7, 2)$ (see the definition at the end of this section) is highly distance-balanced (see Table 1) but it can be easily seen that it is not distance-regular.

Let G be a group and let $S \subset G$ be an inverse closed subset (that is $S = S^{-1}$) not containing the identity. Then the *Cayley graph* $\text{Cay}(G; S)$ is defined to be the graph with vertex set G in which $g \in G$ is adjacent to $h \in G$ whenever $g^{-1}h \in S$.

Let $n \geq 3$ be a positive integer, and let $1 \leq k < n/2$. The generalized Petersen graph $GP(n, k)$ is defined to have the following vertex set and edge set:

$$\begin{aligned} V(GP(n, k)) &= \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\}, \\ E(GP(n, k)) &= \{u_i u_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} \mid i \in \mathbb{Z}_n\} \cup \{u_i v_i \mid i \in \mathbb{Z}_n\}. \end{aligned} \quad (1)$$

The edges of the form $u_i u_{i+1}$ are called *outer edges*, edges of the form $v_i v_{i+k}$ are called *inner edges*, and edges of the form $u_i v_i$ are called *spokes*. Note that $GP(n, k)$ is cubic, and that it is bipartite precisely when n is even and k is odd. It is easy to see that $GP(n, k) \cong GP(n, n-k)$. Furthermore, if the multiplicative inverse k^{-1} of k exists in \mathbb{Z}_n , then $GP(n, k) \cong GP(n, k^{-1})$.

In this paper we first study basic properties of ℓ -distance-balanced graphs. We also give examples of these graphs. In Section 3 we study ℓ -distance-balanced graphs with diameter at most 3. In Section 4 we study the ℓ -distance-balanced property of the generalized Petersen graphs.

2 Basic properties and examples

In this section we present some basic properties of ℓ -distance-balanced graphs and give various examples of such graphs. We first state a fairly straightforward but useful observation and its corollary.

Lemma 2.1 *Let Γ be a connected graph and $u, v \in V(\Gamma)$. If some $\alpha \in \text{Aut}(\Gamma)$ interchanges u and v , then the pair u, v is balanced.*

PROOF. This follows from the fact that for any automorphism $\alpha \in \text{Aut}(\Gamma)$ we have $\alpha(W_{uv}) = W_{\alpha(u)\alpha(v)}$, and so in the case that α interchanges u and v we obtain $\alpha(W_{uv}) = W_{vu}$. As α is a bijection we thus get $|W_{uv}| = |W_{vu}|$. ■

Corollary 2.2 *Let Γ be a connected graph such that for each pair of vertices $u, v \in V(\Gamma)$ there exists an automorphism of Γ interchanging u and v . Then Γ is highly distance-balanced.*

Remark 2.3 *A similar result as in Corollary 2.2 is given in [5, Proposition 2.10] but without a proof. It is then stated (in [5, Corollary 2.12]) that this forces every graph Γ , in which for every pair of vertices at distance ℓ there is an automorphism of Γ mapping one to the other, to be ℓ -distance-balanced. This however does not hold, since it would imply that every vertex-transitive graph is highly distance-balanced. But, for example, the generalized Petersen graph $GP(16, 7)$ with diameter 5 is vertex-transitive (as 7^2 is congruent to 1 modulo 16), but it is not 4-distance-balanced (see Table 1).*

Corollary 2.2 for instance implies that the Petersen graph $GP(5, 2)$ is highly distance-balanced since it is distance-transitive. Moreover, the corollary gives rise to infinitely many highly distance-balanced graphs. In particular, every Cayley graph of an abelian group is highly distance-balanced.

Proposition 2.4 *Let A be a finite abelian group and let $S \subset A$ be an inverse closed subset of A not containing the identity 1. If $\langle S \rangle = A$ then the Cayley graph $\text{Cay}(A; S)$ is highly distance-balanced.*

PROOF. Since the graph $\Gamma = \text{Cay}(A; S)$ is vertex transitive it suffices to prove that there exists no $a \in A$ such that the pair $1, a$ is non-balanced. Observe that, since A is abelian, the permutation τ of A , mapping each element to its inverse, is an automorphism of Γ . Namely, for any pair a, as of adjacent vertices of Γ their images a^{-1} and $s^{-1}a^{-1} = a^{-1}s^{-1}$ are adjacent as $S = S^{-1}$. Likewise, for any $a \in A$ the permutation t_a , mapping each $b \in A$ to ab , is clearly an automorphism of Γ . Since the product $t_a\tau$ interchanges the vertices 1 and a , Corollary 2.2 implies that Γ is highly distance-balanced. ■

The above proposition implies that cycles, being Cayley graphs of cyclic groups, are highly distance-balanced. One of the next natural families of graphs that could possibly provide interesting examples and nonexamples of ℓ -distance-balanced graphs is the family of cubic graphs. Within this family the well known family of generalized Petersen graphs might be a good place to start the investigation. As we will see in Section 4 the problem of determining all ℓ such that, for given $n \geq 3$ and $1 \leq k < n/2$, the generalized Petersen graph $GP(n, k)$ is ℓ -distance-balanced, does not seem to be easy. Of course, for some pairs of n and k , the problem is very easy. For instance, the prisms, being Cayley graphs of abelian groups, are not very interesting.

Corollary 2.5 *Let $n \geq 3$ be an integer. Then the prism $GP(n, 1)$ is highly distance-balanced.*

PROOF. Since the generalized Petersen graph $GP(n, 1)$ is isomorphic to the Cayley graph $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2; \{(1, 0), (-1, 0), (0, 1)\})$, it is highly distance-balanced by Proposition 2.4. ■

As we will see in Section 4, the generalized Petersen graphs $GP(n, k)$ with $k \geq 2$ are much more interesting. However, before we turn our attention to these graphs let

us mention a few more interesting examples. The Cayley graph $\text{Cay}(A_4; S)$, where $S = \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2)(3\ 4)\}$ (the truncation of the tetrahedron), which is a cubic graph of diameter 3, is 1-distance-balanced and 3-distance-balanced, but it is not 2-distance-balanced. Namely, the pair $u = \text{id}$ and $v = (1\ 3\ 4)$ is not balanced as we get $|W_{uv}| = 4$ and $|W_{vu}| = 5$ (see Figure 1). Similarly, the Cayley graph $\text{Cay}(S_4; \{(1\ 2), (2\ 4), (1\ 2)(3\ 4)\})$, which is a cubic graph of diameter 4, is 1-distance-balanced and 2-distance-balanced but is not 3-distance-balanced (see Figure 1) nor 4-distance-balanced since none of the pairs $\text{id}, (1\ 4\ 3)$, nor $\text{id}, (1\ 2\ 4\ 3)$ is balanced. It thus seems that already with cubic graphs the situation regarding ℓ -distance-balancedness is quite interesting.

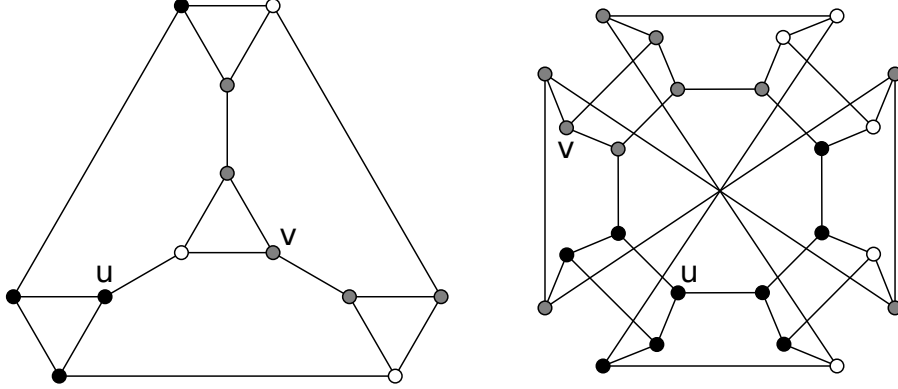


Figure 1: The Cayley graph $\text{Cay}(A_4; \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2)(3\ 4)\})$ is not 2-distance-balanced and the Cayley graph $\text{Cay}(S_4; \{(1\ 2), (2\ 4), (1\ 2)(3\ 4)\})$ is not 3-distance-balanced.

Another interesting family of graphs are distance degree regular graphs (first introduced in [8] and later called strongly distance-balanced graphs in [11]). A connected graph Γ with diameter D is called *distance degree regular*, whenever $|N_i(u)| = |N_i(v)|$ for any two vertices u, v of Γ and for any $0 \leq i \leq D$. It was shown in [11] that every distance degree regular graph is automatically distance-balanced. As we now show, it is also 2-distance-balanced provided it is bipartite.

Theorem 2.6 *Let Γ denote a bipartite distance degree regular graph. Then Γ is 1- and 2-distance-balanced.*

PROOF. By the above remark we only need to show that Γ is 2-distance-balanced. Pick vertices u, v of Γ such that $d(u, v) = 2$. We show that then $|W_{uv}| = |W_{vu}|$. Observe first that since Γ is bipartite we have

$$W_{uv} = \bigcup_{i=1}^{D-1} (N_{i-1}(u) \cap N_{i+1}(v)), \quad W_{vu} = \bigcup_{i=1}^{D-1} (N_{i-1}(v) \cap N_{i+1}(u)). \quad (2)$$

To prove the theorem it thus suffices to verify that for each $1 \leq i \leq D-1$ the equality $|N_{i-1}(u) \cap N_{i+1}(v)| = |N_{i-1}(v) \cap N_{i+1}(u)|$ holds. We show this using induction on i . Obviously, $|N_0(u) \cap N_2(v)| = |N_0(v) \cap N_2(u)| = 1$. Since $d(u, v) = 2$ and Γ is bipartite

we have that $N_1(u) = (N_1(u) \cap N_1(v)) \cup (N_1(u) \cap N_3(v))$ and $N_1(v) = (N_1(u) \cap N_1(v)) \cup (N_1(v) \cap N_3(u))$. Thus, since $|N_1(u)| = |N_1(v)|$ we get $|N_1(u) \cap N_3(v)| = |N_1(v) \cap N_3(u)|$.

Suppose now that for some $2 \leq k \leq D - 2$ we have that $|N_{j-1}(u) \cap N_{j+1}(v)| = |N_{j-1}(v) \cap N_{j+1}(u)|$ for each $1 \leq j \leq k$. Observe that $N_k(u)$ is a disjoint union of $N_k(u) \cap N_{k+2}(v)$, $N_k(u) \cap N_k(v)$ and $N_k(u) \cap N_{k-2}(v)$. Similarly, $N_k(v)$ is a disjoint union of $N_k(v) \cap N_{k+2}(u)$, $N_k(v) \cap N_k(u)$ and $N_k(v) \cap N_{k-2}(u)$. Since $|N_k(u)| = |N_k(v)|$ we thus get

$$|N_k(u) \cap N_{k+2}(v)| + |N_k(u) \cap N_{k-2}(v)| = |N_k(v) \cap N_{k+2}(u)| + |N_k(v) \cap N_{k-2}(u)|.$$

Since, by induction hypothesis, $|N_k(u) \cap N_{k-2}(v)| = |N_k(v) \cap N_{k-2}(u)|$ holds, we thus obtain $|N_k(u) \cap N_{k+2}(v)| = |N_k(v) \cap N_{k+2}(u)|$, which completes the induction step. ■

Corollary 2.7 *Let Γ denote a connected bipartite vertex transitive graph. Then Γ is 2-distance-balanced. In particular, every bipartite connected Cayley graph is 2-distance-balanced.*

PROOF. Observe that every vertex transitive graph is clearly distance degree regular. The result now follows immediately from Theorem 2.6. ■

We remark that the result of Theorem 2.6 (as well as Corollary 2.7) cannot be extended to nonbipartite distance degree regular graphs. Namely, the truncation of the tetrahedron is vertex-transitive (being a Cayley graph) and as such is distance degree regular but is not 2-distance-balanced as was indicated on Figure 1.

3 Graphs of diameter at most 3

In this section we study graphs with diameter 2 or 3 (the graphs of diameter 1, i. e. the complete graphs, are of course 1-distance-balanced). Assume first that Γ has diameter 2. By [10, Corollary 2.3], Γ is 1-distance-balanced if and only if it is regular. It thus remains to determine when Γ is 2-distance-balanced. To do so we first need some more terminology.

Let Γ_1 and Γ_2 be graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 . The *union* $\Gamma = \Gamma_1 \cup \Gamma_2$ of graphs Γ_1 and Γ_2 is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The *join* $\Gamma = \Gamma_1 + \Gamma_2$ of graphs Γ_1 and Γ_2 is the graph $\Gamma_1 \cup \Gamma_2$ together with all the edges joining V_1 and V_2 . Note that the join operation is both commutative and associative. We can thus speak of the graph $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_t$ whenever the graphs Γ_i have pairwise disjoint vertex- and edge-sets. We call the graphs Γ_i ($1 \leq i \leq t$) the *components* of the join Γ . We remark that if a graph Γ is a join of at least two graphs then clearly the diameter of Γ is at most 2.

Theorem 3.1 *Let Γ be a graph with diameter 2. Then Γ is 2-distance-balanced if and only if it is a join of regular graphs, that is $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_t$, where $t \geq 1$ and each of Γ_i is a regular graph.*

PROOF. Suppose first that $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_t$, where each Γ_i , $1 \leq i \leq t$ is a regular graph. By definition of a join of graphs any two vertices from different components Γ_i are adjacent. Thus, if u, v are any two vertices of Γ at distance 2 then there exists some $1 \leq i \leq t$ such that u and v both belong to Γ_i . Since both u and v are adjacent to all the vertices that are not in Γ_i and Γ is of diameter 2, it is clear that W_{uv} consists of u and the neighbours of u in Γ_i , which are not neighbours of v . Similarly, W_{vu} consists of v and the neighbours of v in Γ_i , which are not neighbours of u . Since Γ_i is regular, this implies that Γ is 2-distance-balanced.

Suppose now that Γ is 2-distance-balanced and take an arbitrary pair of vertices u, v of Γ at distance 2. Let $C = N(u) \cap N(v)$. Since Γ is of diameter 2, we get

$$W_{uv} = \{u\} \cup (N(u) \setminus C), \quad W_{vu} = \{v\} \cup (N(v) \setminus C),$$

and so $|W_{uv}| = |W_{vu}|$ implies that u and v have the same valence. Therefore, any two vertices at distance 2 in Γ have the same valence. Let now $k_1 < k_2 < \cdots < k_t$ be all possible degrees of vertices of Γ . If $t = 1$ the graph Γ is regular, so the proof is complete. Suppose then that $t \geq 2$, let $V_i = \{w \in V(\Gamma) \mid |N(w)| = k_i\}$ for all $1 \leq i \leq t$ and let Γ_i denote the subgraph of Γ , induced on V_i . To complete the proof we need to show that any two vertices from different sets V_i are adjacent and that each Γ_i is a regular graph. That the former is true follows from the fact that for $u \in V_i$ and $v \in V_j$ with $1 \leq i < j \leq t$ the valencies of u and v are different, and so they cannot be at distance 2 by the above argument (recall that Γ has diameter 2). That any two vertices from the same set V_i have the same valence within Γ_i is now clear since they have the same valence within Γ and they are both adjacent to all the vertices $w \in V(\Gamma) \setminus V_i$. ■

The following theorem is an immediate corollary of the above theorem and [10, Corollary 2.3].

Theorem 3.2 *Let Γ be a graph with diameter 2. Then Γ is highly distance-balanced if and only if it is regular. Moreover, it is 2-distance-balanced but not 1-distance-balanced if and only if it is a nonregular join of at least two regular graphs.*

Let us point out the following interesting consequence of the above results. For graphs of diameter 2 the fact that the graph in question is 1-distance-balanced implies it is highly distance-balanced. To see that for graphs of larger diameter this does not hold in general it suffices to look at the examples from Figure 1. There we found a 1-distance-balanced graph of diameter 3 which is not 2-distance-balanced and a 1- and 2-distance-balanced graph of diameter 4 that is not 3-distance-balanced. One might think that perhaps a 1-distance-balanced graph Γ is always D -distance-balanced where D is the diameter of Γ . However, since the graph of diameter 4 from Figure 1 is not 4-distance-balanced this also does not hold.

Remark 3.3 *A similar result about 2-distance-balanced graphs of diameter 2 as in Theorem 3.2 was stated in [5, Corollary 2.4]. The authors do not provide a proof and claim it is a corollary of [5, Proposition 2.2], which is supposed to give a necessary and sufficient condition for a graph to be ℓ -distance-balanced. However, the condition is neither*

necessary nor sufficient. For instance, the generalized Petersen graph $GP(13, 3)$ is of diameter 5 and is not 3-distance-balanced but is 4-distance-balanced (see Table 1). However, one can easily check that [5, Proposition 2.2] claims it is 3-distance-balanced but not 4-distance-balanced.

We now turn our attention to graphs of diameter 3. It seems that in this case the general situation is too complicated, so we restrict our consideration to bipartite graphs. Recall that a graph having vertices of two different degrees is called *biregular*.

Proposition 3.4 *Let Γ be a bipartite graph of diameter 3 and bipartition sets X, Y . Then Γ is 1-distance-balanced if and only if it is regular, or it is biregular with $|N(u)| = |Y|/2$ and $|N(v)| = |X|/2$ for every $u \in X$ and $v \in Y$.*

PROOF. Observe that, since Γ is bipartite and of diameter 3, any two vertices of X (or of Y) are at distance 2, and so for all $u \in X$ and $v \in Y$ we have

$$\begin{aligned} N_2(u) &= X \setminus \{u\}, & N_2(v) &= Y \setminus \{v\}, \\ X &= N(v) \cup N_3(v) & \text{and} & \quad Y = N(u) \cup N_3(u). \end{aligned} \quad (3)$$

Let now $u \in X$ and $v \in Y$ be adjacent vertices of Γ . Since Γ is bipartite of diameter 3, the sets W_{uv} and W_{vu} are given by

$$\begin{aligned} W_{uv} &= \{u\} \cup (N(u) \cap N_2(v)) \cup (N_2(u) \cap N_3(v)), \\ W_{vu} &= \{v\} \cup (N(v) \cap N_2(u)) \cup (N_2(v) \cap N_3(u)). \end{aligned} \quad (4)$$

By (3) we have that $N(u) \cap N_2(v) = N(u) \setminus \{v\}$ and $N(v) \cap N_2(u) = N(v) \setminus \{u\}$. Moreover, $N_2(u) \cap N_3(v) = X \setminus N(v)$ and $N_2(v) \cap N_3(u) = Y \setminus N(u)$. It is thus clear that the pair u, v is balanced if and only if $|N(u)| - 1 + |X| - |N(v)| = |N(v)| - 1 + |Y| - |N(u)|$ that is

$$2|N(u)| + |X| = 2|N(v)| + |Y|. \quad (5)$$

Observe that this equality is equivalent both to

$$|N(v)| = |N(u)| + \frac{|X| - |Y|}{2}, \quad \text{and} \quad |N(u)| = |N(v)| + \frac{|Y| - |X|}{2}. \quad (6)$$

We are now ready to finally prove the proposition. Suppose first that Γ is 1-distance-balanced. Then for any pair of adjacent vertices $u \in X$ and $v \in Y$ the equalities (6) hold, and so any two vertices of X , sharing a common neighbor (in Y), have the same degree and likewise any two vertices of Y , sharing a common neighbor (in X), have the same degree. As Γ is connected, all vertices of X have the same degree, say k_X , and all vertices of Y have the same degree, say k_Y . Counting the edges between X and Y in two different ways we obtain the equality

$$k_X|X| = k_Y|Y|.$$

Plugging this into one of the equalities from (6) and multiplying by $|X|$ (or $|Y|$) we get $k_X = |Y|/2$ and $k_Y = |X|/2$, as claimed (note that Γ is regular precisely when $|X| = |Y|$).

To prove the converse suppose first that Γ is regular. Then $|X| = |Y|$, and so (6) implies that a pair of adjacent vertices $u \in X$ and $v \in Y$ is balanced if and only if they

have the same degree which clearly holds since Γ is regular. Suppose finally that Γ is biregular with $|N(u)| = |Y|/2$ and $|N(v)| = |X|/2$ for any $u \in X$ and $v \in Y$. It is now clear that the equalities (6) hold for any pair of adjacent vertices $u \in X$ and $v \in Y$, and so every such pair of vertices is balanced. This shows that Γ is 1-distance-balanced. ■

Proposition 3.5 *Let Γ be a bipartite graph of diameter 3. Then Γ is 2-distance-balanced if and only if the vertices from the same bipartition set have the same degree.*

PROOF. Observe first that since Γ is bipartite of diameter 3 a pair of distinct vertices u, v of Γ is at distance 2 if and only if they both belong to the same bipartition set. Moreover, for any such pair of vertices we have

$$W_{uv} = \{u\} \cup (N(u) \cap N_3(v)), \quad W_{vu} = \{v\} \cup (N(v) \cap N_3(u)) \quad \text{and} \\ N(u) = (N(u) \cap N(v)) \cup (N(u) \cap N_3(v)), \quad N(v) = (N(v) \cap N(u)) \cup (N(v) \cap N_3(u)).$$

Thus the pair u, v is balanced if and only if

$$|N(u)| - |N(u) \cap N(v)| = |N(v)| - |N(v) \cap N(u)|,$$

which is equivalent to u and v being of the same degree. The graph Γ is thus 2-distance-balanced if and only if any two vertices from the same bipartition set have the same degree, which completes the proof. ■

Proposition 3.6 *Let Γ be a bipartite graph of diameter 3 with bipartition sets X, Y . Then Γ is 3-distance-balanced if and only if for any pair of vertices $u \in X$ and $v \in Y$ at distance 3 we have $2|N(u)| + |X| = 2|N(v)| + |Y|$.*

PROOF. Let u, v be vertices of Γ with $d(u, v) = 3$. Then u and v belong to different bipartition sets, and so we may assume $u \in X$ and $v \in Y$. Recall that (3) holds, and so

$$W_{uv} = \{u\} \cup N(u) \cup (X \setminus (\{u\} \cup N(v))) \quad \text{and} \quad W_{vu} = \{v\} \cup N(v) \cup (Y \setminus (\{v\} \cup N(u))).$$

The result follows. ■

Combining the above three results we obtain the following corollary and theorem.

Corollary 3.7 *Let Γ be a bipartite graph of diameter 3 with the bipartition sets X and Y . Then Γ is highly distance-balanced if and only if it is either regular or biregular with each $u \in X$ and $v \in Y$ being of degree $|Y|/2$ and $|X|/2$, respectively.*

Theorem 3.8 *Let Γ be a bipartite graph of diameter 3 with bipartition sets X and Y . Then precisely one of the following holds:*

- (i) Γ is highly distance-balanced.
- (ii) Γ is 2-distance-balanced but not 1-distance-balanced nor 3-distance-balanced.
- (iii) Γ is 3-distance-balanced but not 1-distance-balanced nor 2-distance-balanced.
- (iv) Γ is not ℓ -distance-balanced for any $1 \leq \ell \leq 3$.

PROOF. Suppose first that Γ is 1-distance-balanced. Then Propositions 3.4, 3.5 and 3.6 imply that Γ is both 2-distance-balanced and 3-distance-balanced.

Suppose next that Γ is not 1-distance-balanced but is 2-distance-balanced and 3-distance-balanced. By Propositions 3.4 and 3.5 the graph Γ is not regular, but is biregular with the degree of each $u \in X$ being k_X and the degree of each $v \in Y$ being k_Y . Consequently

$$|X|k_X = |Y|k_Y, \quad \text{and so} \quad k_X = k_Y \frac{|Y|}{|X|}. \quad (7)$$

Take now $u \in X$ and $v \in Y$ such that $d(u, v) = 3$. By Proposition 3.6 we have $2k_X + |X| = 2k_Y + |Y|$. Therefore (7) implies

$$2k_Y(|Y| - |X|) = |X|(|Y| - |X|).$$

Since Γ is not regular, $|X| \neq |Y|$, and so $k_Y = |X|/2$. Similarly we obtain $k_X = |Y|/2$. By Proposition 3.4 the graph Γ is 1-distance-balanced, a contradiction. ■

We remark that each of the four possibilities from the above theorem can indeed occur. Every regular bipartite graph of diameter 3 (for instance, the cube graph) is highly distance-balanced, proving that item (i) is possible. Take any bipartite graph with bipartition sets X and Y of cardinalities 6 and 9, respectively, and where each $u \in X$ has degree 6 and each $v \in Y$ has degree 4. It is easy to see that such graphs exist, have diameter 3 and are 2-distance-balanced but not 3-distance-balanced, proving that item (ii) is possible. Finally, the path of length 3 is clearly a 3-distance-balanced bipartite graph of diameter 3 which is neither 2-distance-balanced nor 1-distance-balanced, and so item (iii) is also possible.

By Corollary 3.7 a regular bipartite graph of diameter 3 is highly distance-balanced. However, if we drop the condition on bipartiteness the result no longer holds. For instance, the truncation of the tetrahedron from Figure 1 which is of course regular (being a Cayley graph) is of diameter 3 but is not 3-distance-balanced. This graph is 2-distance-balanced though. However, also this need not be the case in general. For instance, the Cayley graph $\text{Cay}(D_9; \{t, tr^2, tr^3, r^3, r^6\})$, where $D_9 = \langle t, r \mid t^2, r^9, (tr)^2 \rangle$ is of diameter 3 but is not 2-distance-balanced (the pair $1, r$ is not balanced) nor 3-distance-balanced (the pair $1, r^4$ is not balanced), as can be seen on Figure 2.

4 The ℓ -distance-balanced property of generalized Petersen graphs

As mentioned in Section 2 the problem of determining all ℓ such that a given graph is ℓ -distance-balanced does not seem to be easy even for cubic graphs. To indicate that this might be true we investigate the well known generalized Petersen graphs and their ℓ -distance-balancedness in this section.

We first make the following easy but useful observation.

Corollary 4.1 *Let $n \geq 3$ and $1 \leq k < n/2$ be integers. If the generalized Petersen graph $\Gamma = GP(n, k)$ is not ℓ -distance-balanced for some $1 \leq \ell \leq D$, where D is the diameter*

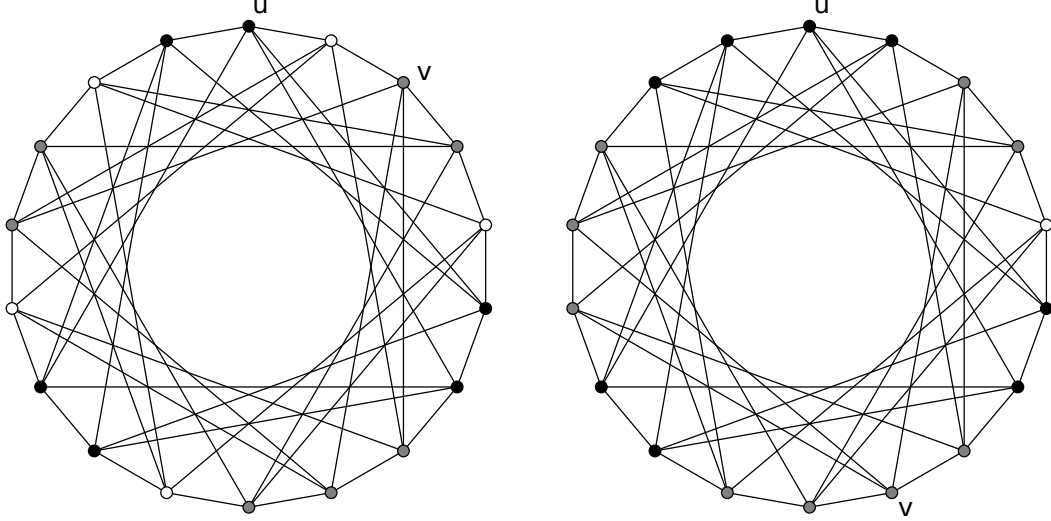


Figure 2: The Cayley graph $\text{Cay}(D_9; \{t, tr^2, tr^3, r^3, r^6\})$ is of diameter 3 but is not 2-distance-balanced nor 3-distance-balanced.

of Γ , then there exists $j \in \mathbb{Z}_n$ such that $d(u_0, v_j) = \ell$ and there is no automorphism of Γ interchanging u_0 and v_j .

PROOF. Observe first that the permutations $\rho = (u_0, u_1, \dots, u_{n-1})(v_0, v_1, \dots, v_{n-1})$ and τ , where $\tau(u_i) = u_{-i}$ and $\tau(v_i) = v_{-i}$ for all $i \in \mathbb{Z}_n$, are automorphisms of Γ . By Lemma 2.1 it thus follows that each pair u_i, u_j and each pair v_i, v_j is balanced. The result now follows immediately from Lemma 2.1. ■

Using Corollary 4.1 and a suitable software package such as MAGMA [2] one may now easily compute all the values ℓ for which a given $GP(n, k)$ is ℓ -distance-balanced. In Table 1 for each pair (n, k) where $5 \leq n \leq 25$ and $2 \leq k < n/2$ the diameter D of $\Gamma = GP(n, k)$ and the set of all $1 \leq \ell \leq D$ for which Γ is ℓ -distance-balanced is given (under the column ℓ -dist. bal.). The possibility of $k = 1$ is omitted in view of Corollary 2.5 (which is also why we start with $n = 5$). We remark that even though $GP(n, k) \cong GP(n, k^{-1})$ (or $GP(n, -k^{-1})$ if $k^{-1} > n/2$) when k is coprime to n we put both possibilities in the table since one might want to search for patterns just based on the value of k .

One of the first things to notice is that each $GP(n, k)$ seems to be D -distance-balanced where D is the diameter of $GP(n, k)$. In view of Corollary 4.1 it would suffice to prove that in $GP(n, k)$, where $k \geq 2$, a pair of vertices at diametral distance is always of the form u_i, u_j or v_i, v_j . Unfortunately, this is not the case in general. For instance the graph $GP(7, 2)$ is of diameter 3 but $d(u_0, v_3) = 3$. Nevertheless it does seem that there are not too many pairs (n, k) such that in $GP(n, k)$ there exists some v_j at diametral distance from u_0 . In fact, a computer search suggests the following might be true.

Conjecture 4.2 *Let $n \geq 3$ and $2 \leq k < n/2$ be integers. If there exists $j \in \mathbb{Z}_n$ such that $d(u_0, v_j) = D$, where D is the diameter of $GP(n, k)$, then either $n = 4m$ and $k = 2m - 1$ for some $m \geq 3$ or the pair (n, k) is one of $(5, 2)$, $(7, 2)$ and $(7, 3)$.*

(n, k)	D	$\ell - \text{dist. bal.}$	(n, k)	D	$\ell - \text{dist. bal.}$	(n, k)	D	par
(5, 2)	2	{1, 2}	(6, 2)	4	{4}	(7, 2)	3	{1, 2, 3}
(7, 3)	3	{1, 2, 3}	(8, 2)	4	{4}	(8, 3)	4	{1, 2, 3, 4}
(9, 2)	4	{3, 4}	(9, 3)	4	{4}	(9, 4)	4	{3, 4}
(10, 2)	5	{1, 2, 3, 4, 5}	(10, 3)	5	{1, 2, 3, 4, 5}	(10, 4)	4	{4}
(11, 2)	5	{4, 5}	(11, 3)	4	{4}	(11, 4)	4	{4}
(11, 5)	5	{4, 5}	(12, 2)	5	{5}	(12, 3)	5	{5}
(12, 4)	5	{5}	(12, 5)	4	{1, 2, 3, 4}	(13, 2)	5	{5}
(13, 3)	5	{4, 5}	(13, 4)	5	{4, 5}	(13, 5)	4	{1, 2, 3, 4}
(13, 6)	5	{5}	(14, 2)	6	{6}	(14, 3)	5	{5}
(14, 4)	5	{1, 2, 3, 4, 5}	(14, 5)	5	{5}	(14, 6)	5	{5}
(15, 2)	6	{6}	(15, 3)	5	{1, 4, 5}	(15, 4)	5	{1, 2, 3, 4, 5}
(15, 5)	5	{5}	(15, 6)	5	{5}	(15, 7)	6	{6}
(16, 2)	6	{6}	(16, 3)	6	{5, 6}	(16, 4)	5	{5}
(16, 5)	6	{5, 6}	(16, 6)	5	{3, 4, 5}	(16, 7)	5	{1, 2, 3, 5}
(17, 2)	6	{6}	(17, 3)	5	{5}	(17, 4)	5	{1, 2, 3, 4, 5}
(17, 5)	5	{1, 2, 3, 4, 5}	(17, 6)	5	{5}	(17, 7)	5	{1, 2, 3, 4, 5}
(17, 8)	6	{6}	(18, 2)	7	{7}	(18, 3)	6	{6}
(18, 4)	5	{5}	(18, 5)	5	{1, 2, 3, 4, 5}	(18, 6)	6	{6}
(18, 7)	5	{1, 2, 3, 4, 5}	(18, 8)	6	{5, 6}	(19, 2)	7	{7}
(19, 3)	6	{6}	(19, 4)	5	{5}	(19, 5)	5	{5}
(19, 6)	6	{6}	(19, 7)	5	{4, 5}	(19, 8)	5	{4, 5}
(19, 9)	7	{7}	(20, 2)	7	{7}	(20, 3)	6	{6}
(20, 4)	6	{6}	(20, 5)	6	{6}	(20, 6)	6	{4, 5, 6}
(20, 7)	6	{6}	(20, 8)	5	{5}	(20, 9)	6	{1, 2, 3, 5, 6}
(21, 2)	7	{7}	(21, 3)	6	{6}	(21, 4)	6	{2, 5, 6}
(21, 5)	6	{2, 5, 6}	(21, 6)	5	{5}	(21, 7)	6	{6}
(21, 8)	5	{1, 2, 5}	(21, 9)	6	{2, 5, 6}	(21, 10)	7	{7}
(22, 2)	8	{8}	(22, 3)	7	{7}	(22, 4)	6	{5, 6}
(22, 5)	6	{1, 2, 3, 4, 5, 6}	(22, 6)	5	{5}	(22, 7)	7	{7}
(22, 8)	6	{5, 6}	(22, 9)	6	{1, 2, 3, 4, 5, 6}	(22, 10)	7	{5, 6, 7}
(23, 2)	8	{8}	(23, 3)	6	{6}	(23, 4)	6	{6}
(23, 5)	5	{5}	(23, 6)	6	{6}	(23, 7)	6	{5, 6}
(23, 8)	6	{6}	(23, 9)	5	{5}	(23, 10)	6	{5, 6}
(23, 11)	8	{8}	(24, 2)	8	{8}	(24, 3)	7	{7}
(24, 4)	6	{1, 6}	(24, 5)	6	{1, 2, 3, 4, 5, 6}	(24, 6)	6	{6}
(24, 7)	6	{1, 2, 3, 4, 5, 6}	(24, 8)	7	{7}	(24, 9)	6	{5, 6}
(24, 10)	6	{5, 6}	(24, 11)	7	{1, 2, 3, 5, 6, 7}	(25, 2)	8	{8}
(25, 3)	7	{7}	(25, 4)	6	{6}	(25, 5)	6	{6}
(25, 6)	6	{6}	(25, 7)	5	{1, 2, 3, 4, 5}	(25, 8)	7	{7}
(25, 9)	6	{6}	(25, 10)	6	{6}	(25, 11)	6	{6}
(25, 12)	8	{8}						

Table 1: The ℓ -distance-balanced property of generalized Petersen graphs.

The reason why we were not able to prove this conjecture in general (we prove that it holds for $k = 2$ in the proof of Theorem 4.7) might be that the diameter and consequently the vertices at diametral distance in $GP(n, k)$ heavily depend on the value of k . In fact, to the best of our knowledge, the diameter of the graphs $GP(n, k)$ is not known in general. However, if Conjecture 4.2 does hold, then the phenomenon observed in Table 1 regarding the diameter does hold in general.

Proposition 4.3 *Suppose that Conjecture 4.2 holds, let $n \geq 3$ and $1 \leq k < n/2$ be integers, and let D be the diameter of the generalized Petersen graph $\Gamma = GP(n, k)$. Then Γ is D -distance-balanced.*

PROOF. By Corollary 2.5 we can assume $k > 1$. Moreover, by Corollary 4.1 we can assume there exists $j \in \mathbb{Z}_n$ such that $d(u_0, v_j) = D$, and by assumption that Conjecture 4.2 holds, we have that $n = 4m$ and $k = 2m - 1$ for some $m \geq 3$ or the pair (n, k) is one of the pairs $(5, 2)$, $(7, 2)$, $(7, 3)$. It is straightforward to check that the graphs $GP(5, 2)$ and $GP(7, 2) \cong GP(7, 3)$ are in fact highly distance-balanced (see also Table 1).

For the rest of the proof we will thus assume that $n = 4m$ and $k = 2m - 1$ for some $m \geq 3$. Since $2k \equiv -2 \pmod{n}$ it is easy to see that for any $0 \leq j \leq 2m$ we have

$$d(u_0, v_j) = \begin{cases} j + 1 & ; \quad 0 \leq j \leq m, \\ 2m - j + 1 & ; \quad m \leq j < 2m, \\ 3 & ; \quad j = 2m. \end{cases}$$

Therefore, if some vertex v_j exists, such that $d(u_0, v_j) = D$, it must be that $v_j = v_m$ (or v_{-m}). Since $(2m - 1)^2 \equiv 1 \pmod{4m}$ it is clear that the permutation σ of $V(\Gamma)$, mapping each u_i to $v_{(2m-1)i}$ and each v_i to $u_{(2m-1)i}$ is an automorphism of Γ . Clearly $\sigma(v_m) = u_{2m^2-m}$ which is either u_{-m} or u_m , depending on whether m is even or odd, respectively. Thus either $\rho^m \sigma$ or $\rho^m \tau \sigma$ interchanges u_0 and v_m , and so this pair of vertices is balanced by Lemma 2.1. It follows that Γ is D -distance-balanced. ■

The data from Table 1 can easily be extended up to at least $n = 200$. The results seem to indicate that for a fixed k there exists some (smallest) integer n_k such that for all $n > n_k$ the graph $GP(n, k)$ is D -distance-balanced but is not ℓ -distance-balanced for any $1 \leq \ell < D$, where D is the diameter of $GP(n, k)$. For instance, it seems that $n_2 = 11$, $n_3 = 16$, $n_4 = 24$, $n_5 = 36$, $n_6 = 48$, $n_7 = 64$, $n_8 = 80$, $n_9 = 100$, $n_{10} = 120$, etc. We therefore make the following conjecture.

Conjecture 4.4 *Let $k \geq 2$ be an integer and let*

$$n_k = \begin{cases} 11 & ; \quad k = 2, \\ (k + 1)^2 & ; \quad k \text{ odd}, \\ k(k + 2) & ; \quad k \geq 4 \text{ even}. \end{cases}$$

Then for any $n > n_k$ the graph $GP(n, k)$ is not ℓ -distance-balanced for any $1 \leq \ell < D$, where D is the diameter of $GP(n, k)$. Moreover, n_k is the smallest integer with this property.

We remark that a result about 1-distance-balancedness of the graphs $GP(n, k)$, related to Conjecture 4.4, was proved in [15]. In particular, it was proved that for any integer $k \geq 2$ and $n > 6k^2$ the graph $GP(n, k)$ is not 1-distance-balanced (see [15, Theorem 2]).

In the reminder of this section we prove that Conjecture 4.4 does hold at least for $k = 2$. We first determine all 1-distance-balanced $GP(n, 2)$ graphs, then all 2-distance-balanced ones and finally all ℓ -distance-balanced ones for $\ell \geq 3$.

For the rest of this section let $\Gamma = GP(n, 2)$ for some $n \geq 5$. We first make the following observations regarding the distances in Γ . Let $0 \leq i \leq n/2$ and consider a shortest path between u_0 and v_i . Clearly such a path contains just one spoke and at most one outer edge. Observe also that $d(u_0, v_i) = d(v_0, u_i)$ (using the automorphisms ρ and τ). It is thus clear that

$$d(u_0, v_i) = d(v_0, u_i) = \begin{cases} 1 + \frac{i}{2} & ; \quad i \text{ even,} \\ 2 + \frac{i-1}{2} & ; \quad i \text{ odd.} \end{cases}$$

This enables us to easily calculate the distances between any pair of vertices of Γ . For instance, if for some $0 \leq i \leq n/2$ every shortest path from u_0 to u_i uses at least one inner edge (which clearly occurs if and only if $n \geq 12$ and $i \geq 6$) then we can assume the first edge of such a path is u_0v_0 , and so $d(u_0, u_i) = d(v_0, u_i) - 1$. We also point out that when n is even, every shortest path from v_0 to v_i with i even uses only inner edges and is thus of length $i/2$, while every shortest path from v_0 to v_i with i odd uses one outer edge and is thus of length $3 + (i-1)/2$. In the case that n is odd, the situation is somewhat different. Namely, in this case, even though one of $(n-1)/2$ and $(n-3)/2$ is odd the shortest path from v_0 to the corresponding v_i uses only inner edges (with the first edge being v_0v_{-2}), and so this v_i is not closer to u_0 than to v_0 . All this enables us to determine all 1-distance-balanced generalized Petersen graphs of the form $GP(n, 2)$.

Proposition 4.5 *Let $n \geq 5$ be an integer. Then the generalized Petersen graph $GP(n, 2)$ is 1-distance-balanced if and only if $n \in \{5, 7, 10\}$.*

PROOF. In view of the automorphisms ρ and τ , Lemma 2.1 implies that the graph $\Gamma = GP(n, 2)$ is 1-distance-balanced if and only if the pair u_0, v_0 is balanced. Using the remarks on distances in Γ one can easily determine the sets $W_{u_0v_0}$ and $W_{v_0u_0}$ and thus complete the proof. For instance, if n is even then clearly all of the vertices v_i with i even are in $W_{v_0u_0}$ while all of the vertices v_i with i odd are in $W_{u_0v_0}$, and so precisely half of the vertices v_i are in $W_{u_0v_0}$ while the other half is in $W_{v_0u_0}$. By the above remarks $u_i \in W_{u_0v_0}$ if and only if $i \in \{0, 1, -1\}$ while all the vertices u_i and u_{-i} for $4 \leq i \leq n/2$ are in $W_{v_0u_0}$. Thus the pair u_0, v_0 is balanced if and only if $3 = n - 7$ that is $n = 10$. The case when n is odd requires a bit more work but can also be done in a similar way.

One can first easily check the graphs $GP(5, 2)$, $GP(7, 2)$ and $GP(9, 2)$ by hand (see also Table 1) to verify that out of the three precisely $GP(5, 2)$ and $GP(7, 2)$ are 1-distance-balanced. To complete the proof we thus only need to show that if $n \geq 11$ is odd the pair u_0, v_0 is not balanced. As was already pointed out we have $u_i \in W_{u_0v_0}$ if and only if $i \in \{0, 1, -1\}$ and since u_2 and u_3 are clearly both at equal distances from u_0 and v_0 we thus also get $u_i \in W_{v_0u_0}$ if and only if $4 \leq i \leq n-4$. Moreover, $W_{v_0u_0}$ contains at least

all of the vertices v_i and v_{-i} for $0 \leq i \leq n/2$ even, while $W_{u_0 v_0}$ contains the vertices v_i and v_{-i} only for $1 \leq i < (n-3)/2$ odd. It thus follows that

$$|W_{u_0 v_0}| \leq 3 + \frac{n-3}{2} = \frac{n+3}{2} \quad \text{and} \quad |W_{v_0 u_0}| \geq n-7 + \frac{n-1}{2} = \frac{3n-15}{2}.$$

Since in the case that $n \geq 11$ we have $3n-15 > n+3$ this finally shows that for $n \geq 11$ the pair u_0, v_0 is not balanced. \blacksquare

As it turns out the 1-distance-balanced graphs $GP(n, 2)$ coincide with the 2-distance-balanced ones.

Proposition 4.6 *Let $n \geq 5$ be an integer. Then the generalized Petersen graph $GP(n, 2)$ is 2-distance-balanced if and only if $n \in \{5, 7, 10\}$.*

PROOF. In view of the automorphisms ρ and τ Lemma 2.1 implies that $\Gamma = GP(n, 2)$ is 2-distance-balanced if and only if the pairs u_0, v_1 and u_0, v_2 are both balanced. Again, one can easily check that for $5 \leq n \leq 14$ both pairs are balanced if and only if $n \in \{5, 7, 10\}$ (see also Table 1). For the rest of the proof we thus assume $n \geq 14$.

We show that in this case the pair u_0, v_2 is not balanced. Let $i \in \mathbb{Z}_n$ be such that a shortest path from u_0 to u_i contains at least one inner edge. Then there also exists a shortest path from u_0 to u_i whose second vertex is v_0 . But since v_0 is a neighbor of both u_0 and v_2 , the vertex u_i cannot be closer to u_0 than to v_2 . It is thus clear that $u_i \in W_{u_0 v_2}$ if and only if $i \in \{1, 0, -1, -2, -3\}$ (recall that $n \geq 14$).

Suppose $i \in \mathbb{Z}_n$ is such that $v_i \in W_{u_0 v_2}$. Then clearly either $i = 1$ or $i = n-1-2j$ for some small enough $j \geq 0$. If n is even, then the path $(v_2, u_2, u_3, v_3, v_5, v_7, \dots, v_{n-1-2j})$ is of length $3 + (n-1-2j-3)/2$, and so $2+j < (n+2-2j)/2$ must hold, that is $j < (n-2)/4$. If however n is odd, then the path $(v_2, v_4, \dots, v_{n-1-2j})$ is of length $(n-1-2j-2)/2$, and so $2+j < (n-3-2j)/2$ must hold, that is $j < (n-7)/4$. In any case we thus find that (by $\lfloor a \rfloor$ we denote the largest integer not exceeding a)

$$|W_{u_0 v_0}| \leq 5 + 1 + \left\lfloor \frac{n-3}{4} \right\rfloor + 1 = 6 + \left\lfloor \frac{n+1}{4} \right\rfloor.$$

Similarly we easily see that $W_{v_2 u_0}$ for sure contains all vertices of the form v_{2i} where $i \geq 1$ and $i-1 < (n-2i)/2 + 1$ in case n is even and where $i-1 < (n-1-2i)/2 + 2$ in case n is odd. In any case $v_{2i} \in W_{v_2 u_0}$ for at least all $1 \leq i \leq \lfloor \frac{n+1}{4} \rfloor$. It is also not difficult to see that $u_j \in W_{v_2 u_0}$ for at least all $2 \leq j \leq \frac{n+1}{2}$ (for instance, if n is odd then $u_{2i+1} \in W_{v_2 u_0}$ for $i \geq 1$ whenever $i-1+2 < (n-2i-1)/2 + 2$). Thus

$$|W_{v_2 u_0}| \geq \left\lfloor \frac{n+1}{4} \right\rfloor + \frac{n-1}{2}.$$

For $n \geq 14$ we get $\frac{n-1}{2} > 6$, and so the pair u_0, v_2 is not balanced. \blacksquare

We are now ready to completely settle the question of ℓ -distance-balancedness for the graphs $GP(n, 2)$. As a consequence we confirm Conjecture 4.4 for $k = 2$.

Theorem 4.7 *Let $n \geq 5$, let $\Gamma = GP(n, 2)$ and let D be the diameter of Γ . Then the following holds.*

- (i) Γ is highly distance-balanced if and only if $n \in \{5, 7, 10\}$.
- (ii) Γ is D - and $(D - 1)$ -distance-balanced but not ℓ -distance-balanced for any $1 \leq \ell \leq D - 2$ if and only if $n \in \{9, 11\}$.
- (iii) Γ is D -distance-balanced but not ℓ -distance-balanced for any $1 \leq \ell \leq D - 1$ if and only if $n \notin \{5, 7, 9, 10, 11\}$.

In particular, if $n > 11$, then Γ is D -distance-balanced but is not ℓ -distance-balanced for any $1 \leq \ell \leq D - 1$.

PROOF. As in the previous two proofs the argument is much easier for large values of n , so we verify the cases with small n separately. We can thus verify that the statement of the theorem is true for all $n \leq 12$ (see also Table 1). For the rest of the proof we thus assume $n \geq 13$. Observe that this implies $d(u_0, u_6) = 5$, and so $D \geq 5$.

We first prove that Γ is D -distance-balanced. We establish this by proving that there exists no v_i such that $d(u_0, v_i) = D$. That Γ is D -distance-balanced then follows from Corollary 4.1. Suppose to the contrary that such a vertex v_i exists and assume with no loss of generality that $i \leq n/2$. Since D is the diameter of Γ we have $d(u_0, u_i) \in \{D, D - 1\}$. Now, $d(u_0, v_5) = d(u_0, v_6) = 4 < 5 = d(u_0, u_6)$, and so $i > 6$. Consequently, every shortest path P from u_0 to u_i uses at least one inner edge. But then there clearly must also exist a shortest path from u_0 to u_i , whose last edge is $v_i u_i$. However, this implies that $d(u_0, v_i) < d(u_0, u_i)$, a contradiction, which thus proves that Γ is D -distance-balanced.

To complete the proof we now only need to show that for any $1 \leq \ell < D$ the graph Γ is not ℓ -distance-balanced. Propositions 4.5 and 4.6 show that this is true for $\ell \in \{1, 2\}$, and so we can assume $3 \leq \ell \leq D - 1$. We first show that there exists v_i such that $d(u_0, v_i) = \ell$. Indeed, by the above argument no v_j is at distance D from u_0 so for some $j \leq n/2$ we have that $d(u_0, u_j) = D$ or $d(u_0, v_j) = D$. In any case $d(u_0, v_j) = D - 1$. But a shortest path from u_0 to v_j for sure uses only one spoke and at most one outer edge, and so there is a shortest path P from u_0 to v_j of length $D - 1$ such that except for perhaps the first two vertices all of its vertices are of the form v_i . But then the vertex on P preceding v_j is of the form v_i and is at distance $D - 2$ from u_0 . The one before it is also in $\{v_i : i \in \mathbb{Z}_n\}$ and is at distance $D - 3$ from u_0 , etc.

Let now v_i with $i \leq n/2$ be such that $d(u_0, v_i) = \ell$. Since $\ell \geq 3$ and $d(u_0, v_3) = d(u_0, v_4) = 3$ we can assume that $4 \leq i \leq n/2$. Let now $V_1 = \{u_j : 1 \leq j \leq i - 1\} \cup \{v_j : 1 \leq j \leq i - 1\}$ and $V_2 = \{u_j : i + 1 \leq j \leq n - 1\} \cup \{v_j : i + 1 \leq j \leq n - 1\}$. It is now clear that for any $0 \leq j \leq i$ every shortest path from either u_0 or v_i to either u_j or v_j uses no vertex from V_2 and similarly for any $i \leq j \leq n$ every shortest path from either u_0 or v_i to either u_j or v_j uses no vertex from V_1 . To determine the sets $W_{u_0 v_i}$ and $W_{v_i u_0}$ we can thus separately consider the graph $\Gamma_1 = \Gamma - V_2$, obtained from Γ by deleting all the vertices from V_2 , and the graph $\Gamma_2 = \Gamma - V_1$. The graphs Γ_1 and Γ_2 both have a similar structure. They only differ in their order (which for both is at least 10). It thus suffices to analyze all of the possibilities for Γ_1 . The situation regarding which vertices of Γ_1 are closer to u_0 than to v_i and vice versa is somewhat different for $i \leq 9$ than for $i \geq 10$ when it only depends on the congruence of i modulo 4. We present all of the possibilities for $4 \leq i \leq 9$ on Figure 3 and the possibilities for $i \geq 10$, depending on the congruence of i modulo 4, on Figure 4. We find that in Γ_1 we always have $|W_{v_i u_0}| \geq |W_{u_0 v_i}|$ and

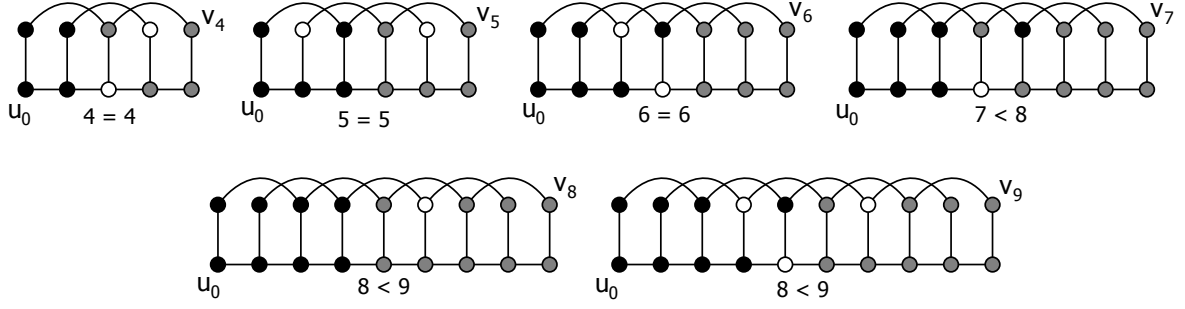


Figure 3: The graphs Γ_1 for $4 \leq i \leq 9$.

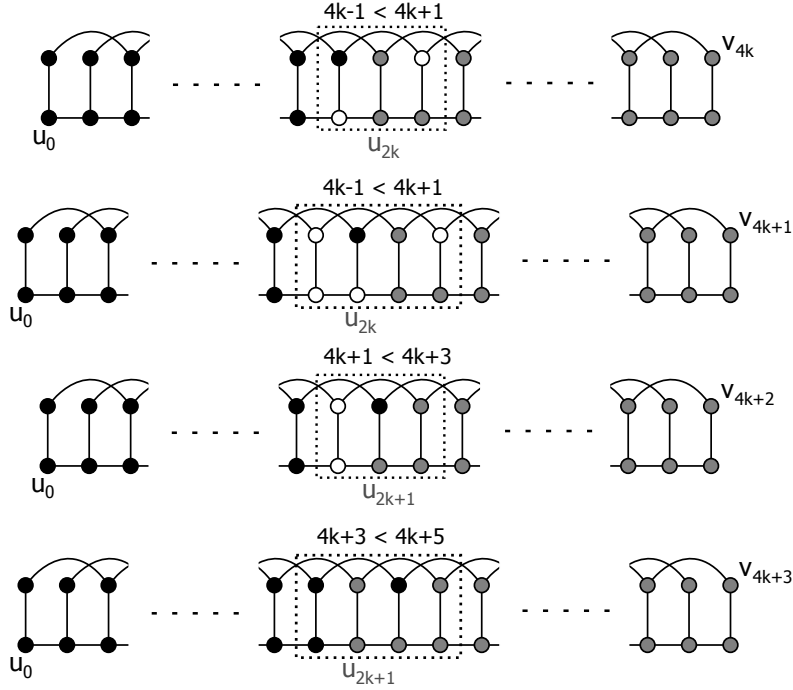


Figure 4: The graphs Γ_1 for $i \geq 10$.

moreover, equality holds only for $4 \leq i \leq 6$. Since a similar situation holds for Γ_2 we see that if either $i \geq 7$ or $n - i \geq 7$ the pair u_0, v_i is not balanced in Γ . But if $i \leq 6$ and $n - i \leq 6$, then $n \leq 12$, a contradiction. Thus the pair u_0, v_i is not balanced in Γ , and so Γ is indeed not ℓ -distance-balanced, as claimed. This completes the proof. ■

5 Suggestions for further research

We conclude the paper with some suggestions for future research. In [7] Handa proved that every 1-distance-balanced graph is 2-connected. In his PhD thesis [6] Frelih proved that this is no longer the case if we move to 2-distance-balanced graphs and characterized connected 2-distance-balanced graphs, which are not 2-connected. We therefore propose

the following problem.

Problem 5.1 *For each $\ell \geq 3$ characterize all connected ℓ -distance-balanced graphs which are not 2-connected.*

In [7] Handa then asked whether all bipartite 1-distance-balanced graphs were also 3-connected. A negative answer to this question was given in [14] with an infinite family of examples. It turned out that even though bipartite 1-distance-balanced graphs which are not 3-connected exist, they have a rather restricted structure. The next problem is thus the following.

Problem 5.2 *Generalize the results of [14] to the class of bipartite ℓ -distance-balanced graphs, $\ell \geq 2$, which are not 3-connected.*

Recently two subfamilies of 1-distance-balanced graphs were introduced, namely the strongly distance-balanced graphs [11] (which coincide with the distance degree regular graphs) and the nicely distance-balanced graphs [13]. These two concepts could easily be extended to ℓ -distance-balanced graphs to obtain strongly ℓ -distance-balanced graphs and nicely ℓ -distance-balanced graphs. We thus propose the following problem.

Problem 5.3 *Introduce the concepts of strongly and nicely ℓ -distance-balanced graphs and investigate the properties of such graphs.*

In the past few years various results describing how the 1-distance-balanced property (and strongly 1-distance-balanced property) of graphs is preserved under various graph products (see for instance [1, 10]). Moreover, in [6] Frelih investigated 2-distance-balanced graphs with respect to Cartesian and lexicographic products. We thus propose to study these things more generally.

Problem 5.4 *Study ℓ -distance-balanced graphs with respect to various graph products.*

In Section 3 the ℓ -distance-balancedness for graphs of diameter at most 3 was investigated where for diameter 3 we restricted ourselves to bipartite graphs. We believe some interesting results could be obtained also for nonbipartite graphs of diameter 3, as well as for bipartite graphs of diameter 4.

Problem 5.5 *Generalize the results of Section 3 to non-bipartite graphs of diameter 3 and to (bipartite) graphs of diameter 4.*

In this paper we also considered the ℓ -distance-balanced property for cubic graphs in some detail. However, as we saw in Section 4 even the generalized Petersen graphs, which are a very special subfamily of cubic graphs, seem to present quite a hard problem when it comes to ℓ -distance-balancedness. The two conjectures from Section 4 (and the related Proposition 4.3) are just two problems regarding the $GP(n, k)$ graphs and their ℓ -distance-balancedness that can be considered. There is at least one other interesting problem regarding these graphs that should be considered. Upon inspection of Table 1 one quickly notices that there are not so many pairs (n, k) for which the graph $GP(n, k)$ is highly distance-balanced but it seems there are infinitely many such pairs. It is thus very natural to consider the following problem.

Problem 5.6 Determine all pairs of integers (n, k) , where $n \geq 5$ and $2 \leq k < n/2$, such that the generalized Petersen graph $GP(n, k)$ is highly distance-balanced or at least determine whether there are infinitely many such pairs.

Recall that we proved in Proposition 2.4 that every connected Cayley graph of an abelian group is highly distance-balanced. As was pointed out in Section 2 this is not true for all Cayley graphs (see the example from Figure 1). However, one might get some similar results for Cayley graphs over groups which are “close” to being abelian, say dihedral groups. We thus propose to study the ℓ -distance-balancedness property of Cayley graphs of dihedral groups.

Problem 5.7 For each connected Cayley graph Γ of a dihedral group determine all $\ell \geq 1$ such that Γ is ℓ -distance-balanced. If this is too difficult in general, consider this problem at least for cubic Cayley graphs of dihedral groups.

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